Preliminary – Concentration bounds

Markov's inequalities

For X a non-negative random variable, for $t > 0$,

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}[X]}{t}.$$

Extension to Markov's inequalities

For X a random variable with mean μ, for $t > 0$,

$$\mathbb{P}(|X-\mu| \geq t) \leq \frac{\mathbb{E}[|X-\mu|]^k}{t^k}.$$

Sub-Gaussian random variables

A random variable X with mean μ is σ-sub-Gaussian if, for all $\lambda \in \mathbb{R}$

$$\mathbb{E}[e^{\lambda(X-\mu)}] \leq e^{\sigma^2\lambda^2/2}.$$

* For $X \sim N(0,1)$, X is 1-sub-Gaussian.

* If X is supported on the interval $[a,b]$, i.e., its support is a subset of the interval $[a,b]$, for some scalars $a<b$, then X is $\frac{b-a}{2}$-sub-Gaussian.

Sub-Gaussian tail bound

For X a σ-sub-Gaussian random variable with mean μ, for $t>0$,

$$\mathbb{P}(X-\mu \geq t) \leq \exp\left(-\frac{t^2}{2\sigma^2}\right).$$

* X_1 and X_2 are independent sub-Gaussian random variables with σ_1 and σ_2, $X_1 + X_2$ is $\sqrt{\sigma_1^2+\sigma_2^2}$-sub-Gaussian.

Hoeffding inequality (sum tail bound)

For X_i, $i=1,\cdots,n$ independent sub-Gaussian random variables with mean μ_i and parameter σ_i, for $t \geq 0$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_i) \geq t\right) \leq \exp\left(-\frac{nt^2}{2\frac{1}{n}\sum_{i=1}^{n}\sigma_n^2}\right)$$

* Let $X_1,\cdots,X_n$ be independent random variables with mean μ_i such that for each $i=1,\cdots,n$ it holds that $a_i \leq X_i \leq b_i$ almost surely, for $a_i, b_i \in \mathbb{R}$, then

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_i) \geq t\right) \leq \exp\left(-\frac{2nt^2}{\frac{1}{n}\sum_{i=1}^{n}(b_i-a_i)^2}\right)$$

Sub-exponential random variables

A random variable X with mean μ is (ν,α)-sub-exponential if, for all $|\lambda| < \frac{1}{\alpha}$,

$$\mathbb{E}[e^{\lambda(X-\mu)}] \le e^{\nu^2\lambda^2/2}.$$

* For $X \sim N(0,1)$, X^2 is $(2,4)$-sub-exponential.

* However, X^2 is not sub-Gaussian with any parameter.

Sub-exponential tail bound

For X a (ν,α)-sub-exponential random variable with mean μ, for $t \ge 0$,

$$\mathbb{P}(X-\mu \ge t) \le \begin{cases} \exp\left(-\frac{t^2}{2\nu^2}\right), & \text{if } 0 \le t \le \frac{\nu^2}{\alpha} \\ \exp\left(-\frac{t}{2\alpha}\right), & \text{if } t > \frac{\nu^2}{\alpha} \end{cases}$$

* For X_i, $i=1,\cdots,n$ independent sub-exponential random variables with parameters (ν_1,α_1) and (ν_2,α_2), X_1+X_2 is $(\sqrt{\nu_1^2+\nu_2^2}, \max(\alpha_1,\alpha_2))$-sub-exponential.

Sum tail bound

For X_i, $i=1,\cdots,n$ independent sub-exponential random variables with mean μ_i and parameters (ν_i,α_i), for $t \ge 0$.

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n (X_i-\mu_i) \ge t\right) \le \begin{cases} \exp\left(-\frac{nt^2}{2\nu_*^2/n}\right), & \text{if } 0 \le t \le \frac{\nu_*^2}{n\alpha_*} \\ \exp\left(-\frac{nt}{2\alpha_*}\right), & \text{if } t > \frac{\nu_*^2}{n\alpha_*} \end{cases},$$

where $\nu_* = \sqrt{\sum_{i=1}^n \nu_i^2}$, $\alpha_* = \max_{i=1,\cdots,n} \alpha_i$.

Bernstein inequality

Bernstein condition

A random variable X with mean μ and variance σ^2 satisfies the Bernstein condition with parameter b if, for $k=2,3,4,\cdots$

$$|\mathbb{E}(X-\mu)^k| \le \frac{1}{2}k!\,\sigma^2 b^{k-2}.$$

* If X is bounded with $|X-\mu| \le b$, then X satisfies the Bernstein condition with parameter b.

* If X satisfies the Bernstein condition with parameter b, then X is sub-exponential with parameters $(\sqrt{2}\sigma, 2b)$.

Bernstein inequality

For X a random variable with mean μ and variance σ^2 that satisfies the Bernstein condition with parameter b, we have, for all $|\lambda| < \frac{1}{b}$, for $t \ge 0$,

$$\mathbb{E}[e^{\lambda(X-\mu)}] \le \exp\left(\frac{\lambda^2\sigma^2/2}{1-b|\lambda|}\right).$$

Furthermore,

$$\mathbb{P}(X-\mu \geq t) \leq \exp\left(-\frac{t^2}{2(\sigma^2+bt)}\right).$$